Review of Electricity and Magnetism
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Applied Math

Make the following definitions before beginning discussion:

$$\begin{aligned}
\mathbf{E} &\equiv \text{the electric field originating from charged particle or distribution} \\
\mathit{\Phi}_{\mathbf{E}}, \mathit{\Phi}_{\mathbf{B}} &\equiv \text{the electric and magnetic flux.} \\
&\quad\ \text{Essentially a measure of the number of field lines passing through a closed surface} \\
\mathbf{J} &\equiv \text{volume current density} \\
\varepsilon &\equiv \text{electromotive force; basically a potential difference}
\end{aligned}$$

1 Electro- and Magnetostatics

Begin with electrostatics. The ultimate goal will be to derive Maxwell's equations for fixed charges (electricity) and steady currents (magnetism). Begin the discussion with the Coulomb's Law, which is;

$$\begin{aligned}
\mathbf{F} &= q_{\text{test}}\mathbf{E} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}.
\end{aligned}$$

Coulomb's law can be used with vector calculus to prove a number of interesting relations. First, let's look at the magnetic flux, which is the measure of the number of field lines passing through a closed surface S, a sphere in this case,

$$\begin{aligned}
\oint_S \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{q}{r^2}\hat{\boldsymbol{r}} \cdot (r^2 \sin\theta d\theta d\phi \hat{\boldsymbol{r}}) \\
&= \frac{q}{\epsilon_0}
\end{aligned}$$

This result is Gauss' Law for a point charge. Applying superposition to the fields of N interacting charges,

$$\mathbf{E} = \sum_{i=1}^{N}\mathbf{E}_i \quad \mathbf{E}_i \equiv i^{\text{th}} \text{ charge electric field contribution}$$

Using linearity of integral operator

$$\begin{aligned}
\oint_S \mathbf{E} \cdot d\mathbf{a} &= \sum_{i=1}^{N}\Big(\oint_S \mathbf{E}_i \cdot d\mathbf{a}\Big) \\
&= \sum_{i=1}^{N}\left(\frac{q_i}{\epsilon_0}\right) = \frac{Q_{\text{enc}}}{\epsilon_0}
\end{aligned}$$

which put into differential form through the divergence theorem is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Now, look at the curl of $\mathbf{E}$; take γ to be any curve along a closed sphere S,

$$\begin{aligned} \oint_\gamma \mathbf{E} \cdot d\boldsymbol{l} &= \frac{1}{4\pi\epsilon_0} \oint_\gamma \frac{q}{r^2} dr \\ &= -\frac{1}{4\pi\epsilon_0} \frac{q}{r} \Big|_{r_a}^{r_b} = 0 \end{aligned}$$

since $r_a = r_b$ for all points on the sphere. This is interpreted as *all points on the sphere have the same potential*. The same result occurs for N interacting charges using the same arguments of superposition and linearity as before. Then

$$\nabla \times \mathbf{E} = \mathbf{0}$$

for all static charge distributions using Stoke's theorem.
Before getting into the divergence and curl of the magnetic field, an interesting aside. First, consider the equation for magnetic force

$$\mathbf{f} = Q(\mathbf{v} \times \mathbf{B})$$

Notice that the force acts in a direction perpendicular to the velocity. The differential work done is

$$dW = \mathbf{f} \cdot d\boldsymbol{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt = 0$$

where $d\boldsymbol{l} = \mathbf{v}dt$ for a charge with magnitude Q moving with velocity $\mathbf{v}$. Think of this moving charge as constituting a current, such as in an electromagnet. This relation states that *the magnetic force does no work*. The magnetic force can change the direction of a charged particle's motion, but not its speed. Now lets move on to Maxwell's equations for magnetostatics. Define volume current density, $\mathbf{J}$ such that

$$\begin{aligned} dI &\equiv \text{current flowing through differential volume} \\ &= \mathbf{J} \cdot d\mathbf{a}. \end{aligned}$$

The concept of volume current density leads to a very nice conservation equation that will be used later to derive Maxwell's equations for electrodynamics. Like most conservation laws, the basic form is: change in volume current density over volume = rate of change of charge density.

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{1}$$

Define Biot-Savart law for determining magnetic field at a point $\mathbf{r}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} dV'$$

where

$$\begin{aligned}\mathbf{r} &\equiv \text{vector from origin to point where field is being evaluated}\\ \mathbf{r}' &\equiv \text{vector from origin to differential volume containing current density}\\ \boldsymbol{\mathfrak{r}} &\equiv \text{vector from differential volume to point where field is being evaluated}\end{aligned}$$

Take the divergence of $\mathbf{B}$

$$\begin{aligned}\nabla \cdot \mathbf{B} &= \frac{\mu_0}{4\pi}\int_{V'} \nabla \cdot \Big(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\Big) dV' \\ &= \frac{\mu_0}{4\pi}\int_{V'} \left(\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \Big(\nabla \times \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\Big)\right) dV' \\ &= 0\end{aligned}$$

Both curl terms go to 0, the first due to $\mathbf{J}$ not being dependent upon the unprimed coordinate system, the second curl equals 0 because of the relation

$$\nabla \times (\mathrm{r}^n \hat{\mathbf{r}}) = 0 \quad \forall n \in \mathbb{Z}$$

This result has an interesting punchline: *there is no such thing as magnetic monopoles.* Magnets exist in nature only with a north and a south pole. That is, in a closed surface, you always enclose 0 net magnetic "charge". Now, let's look at the curl.

The curl of $\mathbf{B}$ is

$$\begin{aligned}\nabla \times \mathbf{B} &= \frac{\mu_0}{4\pi}\int_{V'} \nabla \times \Big(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\Big) dV' \\ &= \frac{\mu_0}{4\pi}\int_{V'} \left(\mathbf{J}\Big(\nabla \cdot \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\Big) - \Big(\mathbf{J} \cdot \nabla\Big)\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\right) dV'\end{aligned}$$

The second dot product term goes to 0 for same reason as in the divergence. Use the relationship

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2} = 4\pi\delta^3(\boldsymbol{\mathfrak{r}})$$

Use the relationship $\boldsymbol{\mathfrak{r}} = \mathbf{r} - \mathbf{r}'$. Integrating over the volume gives

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{r}).$$

The above equation is known as Ampere's law. Now, we have the following, which are known as Maxwell's equations for electro- and magnetostatics

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}\\ \nabla \times \mathbf{E} &= \mathbf{0}\\ \nabla \cdot \mathbf{B} &= 0\\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}(\mathbf{r})\end{aligned}$$

2 Electro- and Magnetodynamics

The electro- and magnetostatic equations equations assume (i) fixed charges and (ii) steady currents $\left(\frac{\partial \rho}{\partial t} = 0\right)$. To adjust for moving charges and non-steady currents, we need to modify the two curl equations. First, the divergence.

Faraday observed that changing the amount of area in a magnetic field (i.e. moving a circular wire in between two poles of an electromagnet) induces a current (potential difference, or voltage). This result is quantified as

$$\varepsilon = -\frac{d\Phi_{\mathbf{B}}}{dt}. \tag{2}$$

Notice, that either the area or the magnetic field could be fluctuating in time to get (2). Let's assume the case where area is constant.

$$\begin{aligned} \varepsilon &= -\frac{d}{dt}\oint_\gamma \mathbf{B} \cdot d\mathbf{a} \\ &= \oint_\gamma -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \end{aligned}$$

Now use the fact that ε is a potential difference. This leads to

$$\varepsilon = \oint_\gamma \mathbf{E} \cdot d\boldsymbol{l} = \oint_\gamma -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

which leads to, using Stokes' theorem,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

This result is Faraday's law in differential form. Now the curl.

So far, what are referred to as Maxwell's equations have no touch of Maxwell. What follows is where Maxwell made his big contribution. First, a simple result of vector calculus is that the divergence of a curl is always 0. Let's see how this applies to our curl equations so far.

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{E}) &= \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0 \ \text{ o.k.} \\ \nabla \cdot (\nabla \times \mathbf{E}) &= \mu_0(\nabla \cdot \mathbf{J}) \ \text{ could be a problem.} \end{aligned}$$

If the current is steady, the above relationship will not violate principles of classical vector calculus. However, if the current is not steady, we can use the conservation relation (1) and Gauss' law

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).$$

Maxwell called the term $\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ the *displacement current* $\mathbf{J}_D$. In order to make the divergence of the curl of $\mathbf{B}$ to be 0, we need to add the displacement current contribution to that of the normal volume current density of Ampere's law. Now, the curl of $\mathbf{B}$ is

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0(\mathbf{J} + \mathbf{J}_D) \\ &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Now we have what are commonly known as the *Maxwell's Equations*

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Now, let's look at Maxwell's equations in a vacuum ($\rho = 0, \mathbf{J} = \mathbf{0}$). Maxwell's equations reduce to the coupled first-order PDEs

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned}$$

We can decouple the equations by applying curl to the curl equations

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \\ \nabla^2 \mathbf{E} &= \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\ &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla \times (\nabla \times \mathbf{B}) &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ \nabla^2 \mathbf{B} &= -\mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \\ &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned}$$

The results above show that the electric and magnetic fields propagate according to the 3D wave equation with speed

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \equiv c.$$

This result also gives a fundamental limit of how fast information can travel in and between reference frames, both inertial and non-inertial. This leads to the principle of relativity.

3 Special Relativity

It had been known for several years that the laws of mechanics were invariant under transformation from one inertial frame to the next. This transformation is known as the Galilean transformation: for two reference frames K, and K' with coordinates (x, y, z, t) and (x', y', z', t'), respectively, and moving with relative velocity $\mathbf{v}$, the space and time coordinates of the two frames are related by

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{v}t \\ t' &= t. \end{aligned}$$

Consider n particles interacting via two-body central potentials. The equation of motion for the i^{th} particle is

$$m_i \frac{d\mathbf{v}_i'}{dt'} = -\nabla_i' \sum_j^n V_{ij}(|\mathbf{x}_i' - \mathbf{x}_j'|).$$

Looking at the Galilean transformation, it is clear that

$$\begin{aligned} \mathbf{v}_i' &= \mathbf{v}_i - \mathbf{v} \\ \nabla_i' &= \nabla_i \\ \frac{d\mathbf{v}_i'}{dt'} &= \frac{d\mathbf{v}_i}{dt} \\ \mathbf{x}_i' - \mathbf{x}_j' &= \mathbf{x}_i - \mathbf{x}_j \end{aligned}$$

and, therefore

$$m_i \frac{d\mathbf{v}_i}{dt} = -\nabla_i \sum_j^n V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|).$$

We have already discussed the wave nature of EM field propagation; is wave phenomena invariant under Galilean transformation? Suppose that a field $\phi(\mathbf{x'}, t)$ satisfies the wave equation.

$$\nabla'^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} = 0$$

in the reference frame K'. Using the definitions of Galilean transformations, the wave equation in K becomes

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\mathbf{v}\cdot\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}\mathbf{v}\cdot\nabla\mathbf{v}\cdot\nabla\right)\phi = 0.$$

Therefore, the wave equation is not Galilean invariant. For sound waves, the lack of invariance is acceptable, so too was it thought for electrodynamics. The preferred reference frame for sound waves is one where the transmitting medium is at rest.. The existence of preferred reference frames where the phenomena are simple is well understood in terms of the bulk motions of the media of propagation. For electromagnetic disturbances, the medium seemed to have no other purpose other than to propagate. This led to three possibilities:

1. The Maxwell equations are incorrect. The proper theory of electromagnetism is Galilean invariant.

2. Galilean relativity applies to classical mechanics, but electromagnetism had a preferred reference frame, the frame in which the lumineferous ether was at rest.

3. There exists a relativity principle for both classical mechanics and electromagnetism, but it was not Galilean relativity. This would imply that the laws of mechanics were in need of modification.

(1) was not likely, given the extraordinary successes of Maxwell's equations in understanding electromagnetic phenomena. (2) was believed to be true, but Michelson-Morley experiment couldn't detect Earth motion relative to ether frame. Therefore, (3) seems most likely.

The transformation needs to preserve the following

$$\begin{aligned} c^2t^2 - (x^2 + y^2 + z^2) &= 0 \\ &= c^2t'^2 - (x'^2 + y'^2 + z'^2) \end{aligned}$$

Therefore

$$c^2t^2 - (x^2 + y^2 + z^2) = \lambda^2[c^2t'^2 - (x'^2 + y'^2 + z'^2)].$$

It is easy to show that $\lambda = 1$. This is the Lorentz transformation and is defined by

$$A = \begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1 + \frac{(\gamma-1)\beta_1^2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \frac{(\gamma-1)\beta_2\beta_1}{\beta^2} & 1 + \frac{(\gamma-1)\beta_2^2}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \frac{(\gamma-1)\beta_3\beta_1}{\beta^2} & \frac{(\gamma-1)\beta_3\beta_2}{\beta^2} & 1 + \frac{(\gamma-1)\beta_3^2}{\beta^2} \end{bmatrix}$$

where

$$\begin{aligned} \beta &= \frac{v}{c} \\ \beta_i &= \frac{v_i}{c} \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}}. \end{aligned}$$

The Lorentz transformation has the nice property

$$\det A = 1$$

There are some nice results that come about from Lorentz transformations. It shows that mass is not Lorentz invariant, but charge is. Time is not invariant, however the the *proper time* τ in the instantaneous rest frame of the system is, and it is defined by

$$\tau = \frac{t}{\gamma}$$

this phenomenon is called *time dilation*. Other such quantities for mass, momentum, energy and length exist. The conclusion is *laws of classical mechanics are not all Lorentz invariant*. This idea forms the basis for the revamping of mechanics. However, the laws of physics are still identical in each inertial reference frame (the postulate of relativity), just how the relationships transform is changed.